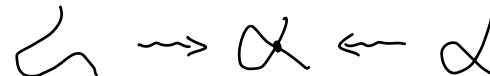


Goal: $\overline{M}_g$ is a sm. proper irred. DM stack of dim $3g-3$ which admits a projective moduli space.

Main steps

- Today {
- ① Algebricity: $\overline{M}_g \subseteq M_g^{an} = \text{stack of all curves}$
 M_g^{an} is an algebraic stack.
 - ② Openness: Stable locus is open $\Rightarrow \overline{M}_g \subseteq M_g^{an}$ open $\Rightarrow$ algebraic
 - ③ Boundedness: $\overline{M}_g$ is of finite type.
 - ④ Existence of coarse moduli space: Keel-Mori } properness.
 - ⑤ Semistable reduction
 - ⑥ projectivity

Def. A curve over a field k is a scheme C of finite type/ k of dim 1

Ex  $C \rightarrow \text{pt} \leftarrow \text{pt} \in W.$

- $g(C) = 1 - \chi(C, \mathcal{O}_C)$
- A family of (n -pointed) curves over a scheme S is a flat, proper, finitely presented $\mathcal{C} \rightarrow S$ of algebraic spaces st. every fiber is a curve (w/ n -sections)

Ex. $\exists$ examples of family of sm. $g=1$ curves where total family is not a scheme.

Prop $\mathcal{C} \rightarrow S$ family of curves Then $\exists S' \xrightarrow{\text{ét}} S$ st. $\mathcal{C}_{S'} \rightarrow S'$ proj.

Pf $s \in S$ $S_n := \text{Spec } \mathcal{O}_{S,s} / \mathfrak{m}_s^{n+1}$, $\hat{S} = \text{Spec } \hat{\mathcal{O}}_{S,s}$.

$$\begin{array}{ccccccc} \mathcal{C}_s & = & \mathcal{C}_0 & \hookrightarrow & \mathcal{C}_1 & \hookrightarrow & \dots \hookrightarrow \hat{\mathcal{C}} \rightarrow \mathcal{C} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \text{Spec}(S) = S_0 & \hookrightarrow & S_1 & \hookrightarrow & \dots \hookrightarrow \hat{S} & \rightarrow & S \end{array}$$

① $\mathcal{C}_s$ = separated 1-dim alg. space $\Rightarrow$ scheme. $\downarrow \Rightarrow$ proj. + proper

② Obstruction to deforming L_{n-1} to $L_n \in H^2(\mathcal{C}_0, \mathcal{O}_{\mathcal{C}_0}) = 0$
also get L_n ample.

③ Case 1: $\hat{S}$ Noeth.

[089N] Grothendieck Existence Thm $\text{Coh}(\hat{\mathcal{C}}) \simeq \varprojlim_n \text{Coh}(\mathcal{C}_n)$

Case 2: S of finite type / $\mathbb{Z}$

Artin approx. to get $(S', s') \rightarrow (S, s)$, L' extending L_0
étale nbhd

Case 3: Noeth approx. [09MV]

□

Prop. $\mathcal{M}_g^{\text{an}}$ is a stack / Schét.

Pf $S' \rightarrow S$ étale cover of schemes st.

$$\begin{array}{ccccc} R = P_1^* \mathcal{C} \xrightarrow{\alpha} P_2^* \mathcal{C} & \xrightarrow{P_1} & \mathcal{C}' & \twoheadrightarrow & \mathcal{C} \\ \downarrow & \searrow P_2 \circ \alpha^{-1} & \downarrow & & \downarrow \\ S'_1 \times_S S' & \hookrightarrow & S' & \rightarrow & S \end{array}$$

w/ α satisfying cocycle cond.

R is an étale equiv relation.

$\mathcal{C} := \mathcal{C}' / R \rightarrow S$.

□

Lem $\mathcal{M}_g^{\text{all}} \xrightarrow{\Delta} \mathcal{M}_g^{\text{an}} \times \mathcal{M}_g^{\text{an}}$ representable

$[\Rightarrow \forall H \xrightarrow{\text{scheme}} \mathcal{M}_g^{\text{an}} \text{ representable}]$

Pf Given $T \xrightarrow{\text{scheme}} \mathcal{M}_g^{\text{an}} \times \mathcal{M}_g^{\text{an}} \rightrightarrows \mathcal{C}_1, \mathcal{C}_2 \rightarrow T$

$$\begin{array}{ccc} \text{Isom}_T(C_1, C_2) & \longrightarrow & T \\ \downarrow & & \downarrow \\ M_g^{\text{an}} & \longrightarrow & M_g^{\text{an}} \times M_g^{\text{an}} \end{array}$$

① Reduce to C_1, C_2 projective.

$T' \xrightarrow{\text{ét}} T$ s.t. C_1, T', C_2, T' proj. / T' .

$$\begin{array}{ccc} \text{If } \begin{array}{c} \text{scheme} \\ \text{alg space} \end{array} \hookrightarrow \text{Isom}_{T'}(C_1, C_2) & \longrightarrow & T' \\ \downarrow & & \downarrow \text{ét} \\ \text{alg space} \hookrightarrow \text{Isom}_T(C_1, C_2) & \longrightarrow & T \end{array}$$

② C_1, C_2 proj. / T .

$$\begin{array}{c} \text{scheme} \rightarrow \text{Isom}_T(C_1, C_2) \subseteq \text{Mor}_T(C_1, C_2) \subseteq \text{Hilb}(C_1 \times_T C_2 / T) \\ \text{scheme} \end{array}$$

$C_1 \rightarrow C_2 \mapsto \text{graph}[C_1 \rightarrow C_1 \times_T C_2]$

Given
Fact: $X \rightarrow Y$ of schemes, $\exists$ open $T^0 \subseteq T$ s.t.
proper $\swarrow$ $\searrow$ proper
 T

$$\forall S \rightarrow T, X_S \subseteq Y_S \Leftrightarrow S \xrightarrow{\downarrow T^0} T$$

Fact $\Rightarrow$ 1st inclusion rep. open imm.

2nd ——— $\left(\begin{array}{l} [Z \subseteq C_1 \times_T C_2] \text{ is in the image} \\ \text{of 2nd inclusion} \\ \Leftrightarrow Z \hookrightarrow C_1 \times_T C_2 \rightarrow C_1 \text{ gives} \\ \text{an isom} \end{array} \right)$

Thm: M_g^{an} is an algebraic stack loc. of fin. type / $\mathbb{Z}$.

Rmk: $M_{g,1}^{\text{an}}$ is the univ. family of M_g^{an} ($M_{g,1}^{\text{an}} \rightarrow M_g^{\text{an}}$).

Pf NTS: $\forall$ proj. curve C/k . $\exists$ representable, sm.

Sch. $\rightarrow U \rightarrow M_g^{an}$ s.t. $[C]$ is in the img.

Choose $C \hookrightarrow \mathbb{P}^N$ s.t. $h^1(C, \mathcal{O}_C(1)) = 0$, P Hilb. poly.

$$H := \text{Hilb}^P(\mathbb{P}^N_{\mathbb{Z}}/\mathbb{Z}) \quad [C] \simeq e_{h_0} \in \mathcal{C}$$

$$\downarrow \quad \downarrow$$

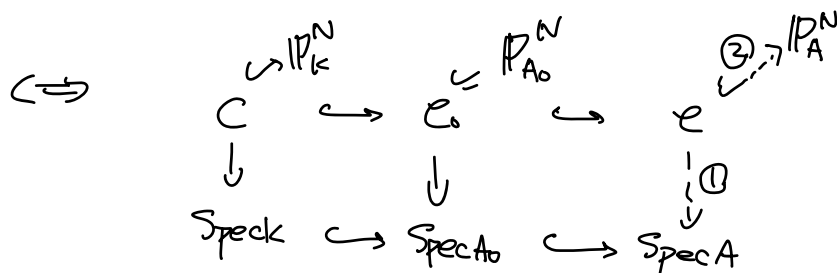
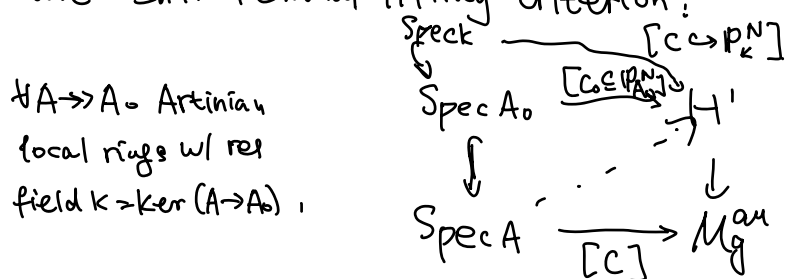
$$h_0 \in H$$

Coh & base change $\Rightarrow \exists h_0 \in H' \subseteq H$ s.t. $\forall h \in H'$, $h^1(e_h, \mathcal{O}_{e_h}(1)) = 0$.

Rep. of $\Delta \Rightarrow H' \rightarrow M_g^{an}$ representable.

Remains to show $H' \rightarrow M_g^{an}$ smooth.

Use Infinitesimal Lifting criterion:



① (D.2.6) $\Rightarrow \text{obs} \in H^2(C, T_C) = 0$

② extend $e_0 \in \mathcal{P}_{A_0}^N$ to $e \in \mathcal{P}_A^N$. $[C|_e]$, def'd by I in $\mathbb{P}_k^N$.

$$\begin{array}{c} \rightarrow \\ \text{Hom}(-, \mathcal{O}_C) \end{array} (0 \rightarrow I/I^2 \rightarrow \Omega_{\mathbb{P}_k^N|_C} \rightarrow \Omega_C \rightarrow 0)$$

$$\leadsto \text{LES: } \text{Hom}_c(I/I^2, \mathcal{O}_c) \rightarrow \text{Ext}_c^1(\Omega_c, \mathcal{O}_c) \rightarrow \text{Ext}_c^1(\Omega_{\mathbb{P}^N|_c}, \mathcal{O}_c)$$

$\begin{array}{ccc} \nearrow & & \nearrow \\ \text{classifies} & [e' \hookrightarrow \mathbb{P}_A^N] \mapsto [e'] & \text{classification} \\ \text{embedded def} & & \text{def. } e_0 \rightarrow \text{Spec } A_0 \\ \text{of } C_0 \subseteq \mathbb{P}_{A_0}^N & & \text{to } e' \rightarrow \text{Spec } A \\ \text{to } e' \in \mathbb{P}_A^N & & \end{array}$

$$H^1(C, T_{\mathbb{P}^N|_c})$$

$$\text{STS: } [e'] \mapsto 0$$

$$0 \rightarrow \mathcal{O}_c \rightarrow \mathcal{O}_c(1)^{\oplus (N+1)} \rightarrow T_{\mathbb{P}^N|_c} \rightarrow 0$$

$$\xrightarrow{\text{LES}} H^1(C, \mathcal{O}_c(1)^{\oplus (N+1)}) \rightarrow H^1(C, T_{\mathbb{P}^N|_c}) \rightarrow H^2(C, \mathcal{O}_c)$$

$\begin{array}{ccc} \parallel & & \parallel \\ 0 & & 0 \end{array}$

□

Thm. $\overline{M}_{g,n}$ is a quasi-compact DM stack sm. over $\text{Spec } \mathbb{Z}$
of rel dim $3g-3+n$, provided $2g-2+n > 0$

Pf • Algebraicity:

$$M_{g,n} \subseteq \overline{M}_{g,n} \xrightarrow{\uparrow \text{Sm. locus}} M_{g,n}^{\text{ss}} \xrightarrow{\uparrow \text{stable locus}} M_{g,n}^{\text{pre}} \xrightarrow{\uparrow \{s \in S \mid \sigma_i(s) \text{ disjoint + sm}\} \text{ open.}} M_{g,n}^{\leq \text{nodal}} \xrightarrow{\uparrow \text{nodal locus is open}} M_g^{\text{all}}$$

• Boundedness:

$(C, p_1, \dots, p_n)$ n -pointed stable curve / k

$\Rightarrow (W_{C/k}(p_1 + \dots + p_n))^{\otimes 3}$ very ample.

$\leadsto C \hookrightarrow \mathbb{P}_k^N$

$\leadsto H \subseteq H_i(b^P(\mathbb{P}^N/\mathbb{Z}) \times (\mathbb{P}^N)^n \xleftarrow{\text{proj. scheme.}}$

loc closed subscheme of $[C \hookrightarrow \mathbb{P}^N, p_1, \dots, p_n] \xleftarrow{C}$

$\leadsto H \hookrightarrow M_{g,n}^{\text{all}} \quad [C \hookrightarrow \mathbb{P}^N, p_1, \dots, p_n] \mapsto (C, p_1, \dots, p_n)$

$$\text{w/ Image}(H) \supseteq \overline{\mathcal{M}}_{g,n}$$

□